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The quantum group of plane motions and basic Bessel functions

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ABSTRACT

Generalised matrix elements of irreducible unitary representations of the quantum group of plane motions are studied. These generalised matrix elements are defined with respect to an orthonormal basis of eigenvectors of a suitable self-adjoint operator which implies a relative bi-invariance property. Using the comultiplication we can derive a q-analogue of Graf's addition formula for q-analogues of the Bessel function which are expressible as a q-hypergeometric ${}_2\varphi_1$-series.

1. INTRODUCTION

The Bessel functions of integer order are closely related to the representation theory of the group of orientation and distance preserving motions of the Euclidean plane, cf. e.g. [20, Chapter 4]. For the quantum group of plane motions it turns out that its representation theory is connected with the so-called Hahn–Exton q-Bessel function, cf. Vaksman and Korogodskiĭ [18], Koelink [5, Chapter 3], [10]. From this interpretation it is possible to prove q-analogues of the Hansen–Lommel orthogonality relations, of the Hankel transform, of the Graf addition formula for the Hahn–Exton q-Bessel function.

In this paper we consider more general matrix elements of the quantum group of plane motions, which can be explicitly determined in some weak sense, i.e. as functionals on the quantised universal enveloping algebra. The general matrix

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elements have relative bi-invariance properties with respect to infinitesimally characterised quantum 'subgroups'. The relative bi-invariance properties are the consequence of the fact that we can determine a basis of eigenvectors for different eigenvalues of a certain self-adjoint operator in $l^2(\mathbf{Z})$. The eigenvectors are described in terms of Jackson's q-Bessel functions, and we find Hansen–Lommel orthogonality relations for Jackson's q-Bessel functions up to a scalar as a consequence.

We also investigate the action of the comultiplication on a generalised matrix element. From the result we derive a q-analogue of Graf's addition formula for q-Bessel functions which are expressible as a q-hypergeometric ${}_2\varphi_1$-series. In [9, § 2] a q-Graf addition formula for q-Bessel functions expressible as a ${}_2\varphi_1$-series was established, see also Kalnins and Miller [4, § 2], but this result is entirely different. The q-Graf addition formula derived in this paper contains some known special cases, and is in fact derived using these known special cases.

The generalised matrix elements in case of the quantum $SU(2)$ group is essentially due to Koornwinder [12], and it has had follow-ups by Noumi and Mimachi [15], [16] and by Koelink [8]. In a remark we show how this approach in the case of the quantum group of plane motions leads to a formal interpretation of Hankel type orthogonality relations which can also be viewed as orthogonality relations for q-Jacobi functions, cf. [17]. These orthogonality relations are closely related to the representation theory of the quantum $SU(1,1)$ group, cf. [14].

This paper can be viewed as a sequel to [10], but the notation and definitions are recalled whenever necessary.

The contents of this paper are as follows. In Section 2 we recall the definition of Jackson's q-Bessel function and we state some results on this function needed in the sequel. The self-adjoint operator and its spectrum are studied in § 3. In § 4 we introduce the generalised matrix elements and we present some of its properties. We also sketch how we can formally interpret the orthogonality relations for the q-Jacobi functions as Hankel type orthogonality relations. Finally, in § 5 we derive a q-analogue of Graf's addition formula.

2. SOME PRELIMINARIES ON q-BESSEL FUNCTIONS

In this section we recall the definition of the Jackson q-Bessel function and we recall some of its properties needed in the sequel of this paper. This q-analogue of the Bessel function was introduced by Jackson in 1903–05, see the references in [3]. The notation and definition for q-hypergeometric series follows the book [2] by Gasper and Rahman. For the sake of completeness we recall the definition of the q-shifted factorials

$$(a;q)_k = \prod_{i=0}^{k-1} (1-aq^i), \quad k \in \mathbf{Z}_+, \qquad (a;q)_\infty = \lim_{k\to\infty} (a;q)_k,$$

$$(a_1,\ldots,a_r;q)_k = (a_1;q)_k \cdots (a_r;q)_k, \quad k \in \mathbf{Z}_+ \cup \{\infty\}$$

assuming from now that $q \in (0,1)$, and of the q-hypergeometric series

$$
{}_r\varphi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right)
= \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_s;q)_k}\,\frac{z^k}{(q;q)_k}\,\left((-1)^k q^{1/2k(k-1)}\right)^{s-r+1}.
$$

In general the radius of convergence is 0, 1 or ∞ corresponding to $r > s+1$, $r = s+1$ or $r < s+1$.

The Jackson q-Bessel function is defined by

$$
(2.1)\qquad J_\alpha^{(2)}(z;q) = \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^\alpha {}_0\varphi_1\left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix};q,-\frac{z^2q^{\alpha+1}}{4}\right).
$$

Use the q-gamma function

$$
(2.2)\qquad \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \to \Gamma(x)
$$

as $q \uparrow 1$ to see that formally $J_\alpha^{(2)}((1-q)x;q)$ tends to the Bessel function

$$
J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\alpha+2k}}{k!\,\Gamma(\alpha+k+1)},
$$

cf. [21, § 3.1(8)].

From the generating function for $J_n^{(2)}(z;q)$, cf. [13, (2.13)], we obtain

$$
(2.3)\qquad J_{-n}^{(2)}(z;q) = (-1)^n J_n^{(2)}(z;q).
$$

Combination with a simple estimate on the ${}_0\varphi_1$-series for $n \in \mathbf{Z}_+$ yields, cf. [7, (3.1)],

$$
(2.4)\qquad |J_n^{(2)}(2z;q)| \le |z|^{|n|}\,\frac{(-|z|^2,-q;q)_\infty}{(q;q)_\infty},\quad \forall n \in \mathbf{Z}.
$$

Jackson's q-Bessel function $J_\alpha^{(2)}(z;q)$ satisfies the three term recurrence relation, cf. [3, (1.18)],

$$
(2.5)\qquad q^\alpha J_{\alpha+1}^{(2)}(z;q) = \frac{2(1-q^\alpha)}{z} J_\alpha^{(2)}(z;q) - J_{\alpha-1}^{(2)}(z;q).
$$

The following q-analogue of Graf's addition formula, cf. (5.2), for the Jackson q-Bessel functions has been proved by Koornwinder and Swarttouw [13, (4.10)]; for $n \in \mathbf{Z}$ and $|y| < 1$

$$
(2.6)\qquad \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} s^k q^{1/2k^2} J_{n+k}^{(2)}(2y;q) J_k^{(2)}(2xq^{-1/2};q) \\ \displaystyle = y^n(-y^2;q)_\infty \frac{(s^{-1}xy^{-1},q^{n+1};q)_\infty}{(q^n s^{-1}xy^{-1},q;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} q^n s^{-1}xy^{-1}, sxy^{-1} \\ q^{n+1} \end{matrix};q,-y^2\right). \end{cases}
$$

3. A SELF-ADJOINT OPERATOR

In this section we study a self-adjoint operator for which there exists an orthonormal basis of eigenvectors for different eigenvalues. The eigenvectors are completely known in terms of Jackson's q-Bessel functions. The choice for the operator under consideration is largely motivated by Koornwinder's [12, §4, Theorem 4.3] approach to the quantum $SU(2)$ group.

First we recall the definition of the quantised universal enveloping algebra of the group of plane motions, cf. [10, § 3], [18, § 1]. The complex associative algebra $\mathcal{U}_q(\mathfrak{m}(2))$ with unit 1 is generated by A, B, C and D subject to the relations

$$(3.1) \qquad AB = qBA, \qquad AC = q^{-1}CA, \qquad AD = 1 = DA, \qquad BC = CB.$$

The algebra $\mathcal{U}_q(\mathfrak{m}(2))$ is an example of Hopf $*$-algebra, cf. [19], but the additional structure will be presented when needed. The $*$-operator is defined by

$$(3.2) \qquad A^* = A, \qquad B^* = C, \qquad C^* = B, \qquad D^* = D.$$

Irreducible $*$-representations of $\mathcal{U}_q(\mathfrak{m}(2))$ are given in [10, § 5], see also [18, § 1]. For $R > 0$ we define the following operators in $l^2(\mathbf{Z})$, equipped with the orthonormal basis $\{e_n\}_{\{n\in\mathbf{Z}\}}$, by giving the action on a basis vector:

$$(3.3) \qquad \begin{cases} t^R(A)e_n = q^n e_n, \qquad t^R(B)e_n = Re_{n+1}, \qquad t^R(C)e_n = Re_{n-1}, \\ t^R(D)e_n = q^{-n}e_n. \end{cases}$$

The operators $t^R(A)$ and $t^R(D)$ are unbounded. As the common domain of the operators $t^R(X)$ for any $X \in \mathcal{U}_q(\mathfrak{m}(2))$ we take $\mathcal{D}(\mathbf{Z})$, the space of finite linear combinations of basis vectors e_n.

Motivated by [12, (4.1)] we introduce

$$(3.4) \qquad X_s = iq^{1/2}B - iq^{-1/2}C + s(A - D) \in \mathcal{U}_q(\mathfrak{m}(2)), \quad s \in \mathbf{R}.$$

From (3.1) and (3.2) we obtain

$$(3.5) \qquad (AX_s^*)^* = AX_s^*,$$

so that the operator $t^R(AX_s^*)$ is symmetric. The operator $t^R(AX_s^*)$ will be considered on the domain

$$\left\{ \sum_{k\in\mathbf{Z}} c_k e_k \in l^2(\mathbf{Z}) \;\middle|\; \sum_{k\in\mathbf{Z}} c_k t^R(AX_s^*)e_k \in l^2(\mathbf{Z}) \right\},$$

which contains $\mathcal{D}(\mathbf{Z})$.

Proposition 3.1. *For $s \neq 0$ the operator $t^R(AX_s^*)$ is self-adjoint and its spectrum is*

$$\{s(q^{2j} - 1)\}_{\{j\in\mathbf{Z}\}} \cup \{-s\}.$$

The eigenspace corresponding to the eigenvalue $s(q^{2j} - 1)$ is one dimensional and spanned by

$$\sum_{k\in\mathbf{Z}} i^{k-j} q^{1/2(k-j)(k-j-1)} J_{k-j}^{(2)}(2Rs^{-1}q^{1/2-j}; q^2) e_k \in l^2(\mathbf{Z}).$$

Proof. The vector $\sum_k c_k e_k$ in $l^2(\mathbf{Z})$ is an eigenvector with eigenvalue ν for the symmetric operator $t^R(AX_s^*)$ if and only if the c_k satisfy the three term recurrence relation

$$(3.6) \qquad iq^{k-1/2}Rc_{k-1} + s(q^{2k}-1)c_k - iq^{k+1/2}Rc_{k+1} = \nu c_k,$$

cf. (3.3). Substitute $c_k = i^{k-j}q^{1/2(k-j)(k-j-1)}d_k$, then we obtain

$$d_{k-1} + q^{2k-2j}d_{k+1} = \frac{q^{-j-1/2}}{R}(\nu - s(q^{2k}-1))d_k.$$

Comparison with (2.5) yields for $\nu = s(q^{2j}-1)$ the solution

$$d_k = J_{k-j}^{(2)}\left(\frac{2R}{s}q^{1/2-j};q^2\right).$$

From the estimate (2.4) it follows that the eigenvectors obtained in this way are in $l^2(\mathbf{Z})$. Since the operator $t^R(AX_s^*)$ is symmetric, we obtain the Hansen–Lommel orthogonality relations [7, Theorem 3.1] for the Jackson q-Bessel function apart from a constant. The completeness of the eigenvectors in $l^2(\mathbf{Z})$ follows from the dual Hansen–Lommel orthogonality relations [7, Theorem 3.3]. (These orthogonality relations will be rephrased in (3.10).)

So we find an orthonormal basis $\{f_j\}_{\{j\in\mathbf{Z}\}}$ of $l^2(\mathbf{Z})$ consisting of eigenvectors for the operator $t^R(AX_s^*)$ with eigenvalues $s(q^{2j}-1)$ and we can reformulate its domain as

$$\mathcal{D}(t^R(AX_s^*)) = \left\{ \sum_{k\in\mathbf{Z}} c_k f_k \in l^2(\mathbf{Z}) \,\middle|\, \sum_{k\in\mathbf{Z}} c_k s(q^{2k}-1)f_k \in l^2(\mathbf{Z})\right\}.$$

Suppose $y = \sum_k w_k f_k \in \mathcal{D}(t^R(AX_s^*)^*) \subset l^2(\mathbf{Z})$, i.e.

$$\langle t^R(AX_s^*)x, y\rangle = \langle x, t^R(AX_s^*)^* y\rangle, \quad \forall x \in \mathcal{D}(t^R(AX_s^*)),$$

then we find $\langle f_k, t^R(AX_s^*)^*y\rangle = s(q^{2k}-1)\overline{w_k}$ for all $k \in \mathbf{Z}$. This implies $y \in \mathcal{D}(t^R(AX_s^*))$ and thus $t^R(AX_s^*)$ is a self-adjoint operator.

Since the spectrum of a self-adjoint operator must be closed, the lemma follows. □

Remark. For $s = 0$ we have $X_0 = iq^{1/2}B - iq^{-1/2}C$, so that $\sum_k c_k e_k$ is an eigenvector of the operator $t^R(AX_0^*)$ for the eigenvalue ν if and only if

$$(3.7) \qquad iq^{k-1/2}Rc_{k-1} - iq^{1/2+k}Rc_{k+1} = \nu c_k, \quad \forall k \in \mathbf{Z}.$$

Substitute $c_k = i^k q^{1/2k(k-1)}d_k$, then we obtain the recurrence relation

$$d_{k-1} + q^{2k}d_{k+1} = \frac{\nu q^{-1/2}}{R}d_k, \quad \forall k \in \mathbf{Z}.$$

A solution for this recurrence relation is given by

$$d_k = {}_1\psi_2\left(\begin{matrix} Rq^{1/2}\nu^{-1} \\ 0,0 \end{matrix};q^2, -\frac{\nu q^{-1/2}}{R}q^{2-2k}\right),$$

which is an absolutely convergent bilateral series, cf. [2, Chapter 5] for the definition of such a bilateral series. This solution can be obtained by assuming that d_k is a Laurent series in q^{2k}. Substitution in the recurrence relation for the d_k yields a simple recurrence relation for the coefficients of this Laurent series. For $\nu = 0$ this series can be summed by Jacobi's triple product identity, cf. [2, § 1.6]. In this case we obtain

$$c_k = i^k q^{1/2k(k-1)}(q^{2k+2}, q^{2-2k}, q^4; q^4)_\infty.$$

The corresponding eigenvector is not an element of $l^2(\mathbf{Z})$ and it is easily seen that any solution of (3.7) for $\nu = 0$ is not an element of $l^2(\mathbf{Z})$. From now on we will ignore the case $s = 0$.

We will define the orthonormal basis of $l^2(\mathbf{Z})$ consisting of eigenvectors of $t^R(AX_s^*)$ explicitly. For non-zero real s we put

$$(3.8) \qquad f^{R,k}(s) = \sum_{n=-\infty}^{\infty} f_n^{R,k}(s) e_n \in l^2(\mathbf{Z})$$

with

$$(3.9) \qquad f_n^{R,k}(s) = \frac{i^{n-k} q^{1/2(n-k)(n-k-1)}}{(-(R^2/s^2)q^{1-2k}; q^2)_\infty^{1/2}} J_{n-k}^{(2)}\left(\frac{2R}{s} q^{1/2-k}; q^2\right).$$

Note that $f_n^{R,k}(\infty) = \delta_{n,k}$, since $J_n^{(2)}(0; q) = \delta_{n,0}$ by (2.1) and (2.3). The Hansen–Lommel orthogonality relations for the Jackson q-Bessel function can be re-written as

$$(3.10) \qquad \sum_{n=-\infty}^{\infty} f_n^{R,k}(t) \overline{f_n^{R,l}(t)} = \delta_{k,l} = \sum_{n=-\infty}^{\infty} f_k^{R,n}(t) \overline{f_l^{R,n}(t)},$$

cf. [7, Theorems 3.1, 3.3].

4. GENERALISED MATRIX ELEMENTS ON THE QUANTUM $M(2)$ GROUP

Generalised matrix elements have been first introduced by Koornwinder [12] for spherical elements on the quantum $SU(2)$ group. Noumi and Mimachi [15], [16] (see also [8]) have extended this approach to more general matrix elements on the quantum $SU(2)$ group. In a remark we discuss how this approach can give some formal results in this case. Here we define the generalised matrix elements in a weak sense as matrix elements of the irreducible $*$-representation t^R of $\mathcal{U}_q(\mathrm{m}(2))$, cf. (3.3), with respect to elements of the orthonormal basis established in the previous section.

We consider the matrix elements of the $*$-representation t^R of $\mathcal{U}_q(\mathrm{m}(2))$ in $l^2(\mathbf{Z})$ with respect to the orthonormal basis $\{f^{R,k}(s)\}_{\{k\in\mathbf{Z}\}}$ constructed in the previous section. Explicitly, we define for $R > 0$, $i, j \in \mathbf{Z}$, $X \in \mathcal{U}_q(\mathrm{m}(2))$, $t, s \in \mathbf{R}\backslash\{0\}$,

$$(4.1) \qquad \langle b_{i,j}^R(t,s), X\rangle = \langle t^R(X) f^{R,j}(s), f^{R,i}(t)\rangle$$

or equivalently as an identity in $(\mathcal{U}_q(\mathrm{m}(2)))^*$

$$(4.2)\qquad b^R_{i,j}(t,s) = \sum_{n,m=-\infty}^{\infty} f^{R,j}_m(s)\overline{f^{R,i}_n(t)}\, t^R_{n,m},$$

where $\langle t^R_{n,m}, X\rangle = t^R_{n,m}(X) = \langle t^R(X)e_m, e_n\rangle$ are the matrix coefficients of the $*$-representation t^R with respect to the standard basis $\{e_n\}_{\{n\in\mathbf{Z}\}}$ of $l^2(\mathbf{Z})$. Note that $b^R_{i,j}(\infty,\infty) = t^R_{i,j}$. The element $b^R_{i,j}(t,s)$ satisfies the following relative bi-invariance property:

$$\begin{aligned}&\langle b^R_{i,j}(t,s), (X_t A)X(AX_s^*)\rangle\\ &\quad = ts(q^{2j}-1)(q^{2i}-1)\langle b^R_{i,j}(t,s), X\rangle, \quad \forall X\in\mathcal{U}_q(\mathfrak{m}(2)).\end{aligned}$$

This follows directly from (4.1), Proposition 3.1, t^R being a $*$-representation of $\mathcal{U}_q(\mathfrak{m}(2))$ and $(X_t A)^* = AX_t^*$.
Note that for a general element a of the form

$$a = \sum_{n,m\in\mathbf{Z}} \gamma_{n,m}\, t^R_{n,m}, \quad \gamma_{n,m}\in\mathbf{C},$$

we find a well-defined element of $(\mathcal{U}_q(\mathfrak{m}(2)))^*$ if

$$(4.3)\qquad \sum_{n\in\mathbf{Z}} |\gamma_{n,n+i}| q^{pn} < \infty, \quad \forall p,i\in\mathbf{Z}.$$

This follows from (3.3) and the fact that $B^kC^lA^p$, $k,l\in\mathbf{Z}_+$, $p\in\mathbf{Z}$, yields a basis for the underlying linear space of $\mathcal{U}_q(\mathfrak{m}(2))$, which follows easily from the commutation relations (3.1). From the estimate (2.4) on the Jackson q-Bessel function together with the explicit form (3.9) for $f^{R,j}_m(s)$ it follows that (4.3) is satisfied for $b^R_{i,j}(t,s)$ as in (4.2). Hence, $b^R_{i,j}(t,s)$ is a well-defined element of $(\mathcal{U}_q(\mathfrak{m}(2)))^*$.

For an element of the form $X = A^\lambda \in \mathcal{U}_q(\mathfrak{m}(2))$, $\lambda\in\mathbf{Z}$, which are the so-called group-like elements of $\mathcal{U}_q(\mathfrak{m}(2))$, we can give a closed expression for $\langle b^R_{i,j}(t,s), X\rangle$.

Proposition 4.1. *For $\lambda\in\mathbf{Z}$, $|R^2q^{1-2j}/s^2| < 1$ we have*

$$(4.4)\qquad \left\{\begin{aligned}&\langle b^R_{i,j}(t,s), A^\lambda\rangle\\ &\quad = i^{i-j}q^{1/2(i-j)(i-j-1)}\left(\frac{R}{s}\right)^{i-j} q^{(i-j)(1/2-j)}\,\frac{(-(R^2/s^2)q^{1-2j};q^2)_\infty^{1/2}}{(-(R^2/t^2)q^{1-2i};q^2)_\infty^{1/2}}\\ &\quad\times q^{i\lambda}\,\frac{((s/t)q^{2-\lambda-2i+2j}, q^{2i-2j+2};q^2)_\infty}{((s/t)q^{2-\lambda}, q^2;q^2)_\infty}\\ &\quad\times {}_2\varphi_1\left(\begin{matrix}(s/t)q^{2-\lambda}, (s/t)q^\lambda\\ q^{2i-2j+2}\end{matrix};q^2, -\frac{R^2}{s^2}q^{1-2j}\right).\end{aligned}\right.$$

Proof. From $\langle t^R_{n,m}, A^\lambda\rangle = \delta_{n,m}q^{n\lambda}$, cf. (3.3), and (3.9) we get an explicit expression for $\langle b^R_{i,j}(t,s), A^\lambda\rangle$ as a single sum over $n\in\mathbf{Z}$. Replace the summation parameter n by $n+i$ to see that

$$\langle b^R_{i,j}(t,s), A^\lambda\rangle = \frac{i^{i-j} q^{1/2(i-j)(i-j-1)} q^{i\lambda}}{(-(R^2/s^2)q^{1-2j};q^2)^{1/2}_\infty(-(R^2/t^2)q^{1-2i};q^2)^{1/2}_\infty}$$
$$\times \sum_{n=-\infty}^{\infty} q^{n^2} q^{n(i-j+\lambda-1)} J^{(2)}_{n+i-j}\left(\frac{2R}{s}q^{1/2-j};q^2\right)$$
$$\times J^{(2)}_n\left(\frac{2R}{t}q^{1/2-i};q^2\right).$$

Compare this expression with the Graf addition formula for the Jackson q-Bessel function (2.6) with q, s, n, x and y replaced by q^2, $q^{i-j+\lambda-1}$, $i-j$, $Rq^{3/2-i}/t$ and $Rq^{1/2-j}/s$ to obtain the statement of the proposition. □

Remark. In case of a negative integer power of q^2 in the lower parameter of the ${}_2\varphi_1$-series in (4.4) the result has to be interpreted according to

$$(q^{1-n};q)_\infty \sum_{k=0}^{\infty} \frac{c_k}{(q^{1-n};q)_k} = \sum_{k=n}^{\infty} c_k(q^{1-n+k};q)_\infty$$
$$= (q^{n+1};q)_\infty \sum_{k=0}^{\infty} \frac{c_{k+n}}{(q^{n+1};q)_k}$$

for $n \in \mathbf{Z}_+$. This formula leads to

$$\frac{(q^{1-n};q)_\infty}{(q;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} a,b \\ q^{1-n}\end{matrix};q,z\right)$$
$$= z^n \frac{(a,b;q)_\infty}{(aq^n,bq^n;q)_\infty}\frac{(q^{n+1};q)_\infty}{(q;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} aq^n,bq^n \\ q^{n+1}\end{matrix};q,z\right)$$

valid for $n \in \mathbf{Z}$. Now use Heine's transformation formula [2, (1.4.6)] to the right hand side to see that for $n \in \mathbf{Z}$

$$\text{(4.5)}\quad \left\{\begin{aligned} &\frac{(q^{1-n};q)_\infty}{(q;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} a,b \\ q^{1-n}\end{matrix};q,z\right) \\ &= z^n \frac{(a,b,q^{n+1},zabq^{n-1};q)_\infty}{(aq^n,bq^n,q,z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} q/a,q/b \\ q^{n+1}\end{matrix};q,abzq^{n-1}\right).\end{aligned}\right.$$

If we apply this result to the right hand side of (4.4) this would yield the same result as shifting the summation parameter n to $n+j$ instead of $n+i$ in the proof of (4.4).

Next we introduce a $*$-operator and a comultiplication on the generalised matrix elements by defining a $*$-operator and comultiplication on linear functionals on $\mathcal{U}_q(\mathfrak{m}(2))$ by, cf. [19],

$$\Delta : (\mathcal{U}_q(\mathfrak{m}(2)))^* \to (\mathcal{U}_q(\mathfrak{m}(2)) \times \mathcal{U}_q(\mathfrak{m}(2)))^*,$$
$$\langle \Delta(a), X \otimes Y\rangle = \langle a, XY\rangle$$

and

$$*:(\mathcal{U}_q(\mathfrak{m}(2)))^* \to (\mathcal{U}_q(\mathfrak{m}(2)))^*, \qquad \langle a^*, X\rangle = \overline{\langle a, S(X)^*\rangle},$$

where $S:\mathcal{U}_q(\mathfrak{m}(2)) \to \mathcal{U}_q(\mathfrak{m}(2))$ is the antipode of the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{m}(2))$, i.e. the linear, antimultiplicative mapping sending A, B, C, D to D, $-q^{-1}B$, $-qC$, A, cf. [10, (3.4)].
The product of two linear functionals $a, b \in (\mathcal{U}_q(\mathfrak{m}(2)))^*$ is defined by

$$\langle ab, X\rangle = \langle a\otimes b, \Delta(X)\rangle,$$

where $\Delta:\mathcal{U}_q(\mathfrak{m}(2)) \to \mathcal{U}_q(\mathfrak{m}(2))\otimes\mathcal{U}_q(\mathfrak{m}(2))$ is the comultiplication of the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{m}(2))$, cf. [10, (3.4)].

The elements $b^R_{i,j}(t,s)$ are called generalised matrix elements and this name is motivated by the following proposition. If we take $s=t=r$ in that proposition we see that the infinite matrix $(b^R_{i,j}(t,t))_{i,j\in\mathbf{Z}}$ possesses properties similar to a unitary matrix corepresentation of a Hopf $*$-algebra, cf. e.g. [11, §§2.3, 3.3] for an explanation of these notions.

Proposition 4.2. *For non-zero real s, t, r we have*

$$\Delta(b^R_{i,j}(t,s)) = \sum_{k=-\infty}^{\infty} b^R_{i,k}(t,r)\otimes b^R_{k,j}(r,s) \tag{4.6}$$

as an identity in $(\mathcal{U}_q(\mathfrak{m}(2))\times\mathcal{U}_q(\mathfrak{m}(2)))^*$ and

$$\begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} b^R_{i,k}(t,s)(b^R_{j,k}(r,s))^* = \langle f^{R,j}(r), f^{R,i}(s)\rangle \\ \displaystyle\qquad\qquad = \sum_{k=-\infty}^{\infty} (b^R_{k,i}(t,s))^* b^R_{k,j}(t,r) \end{cases} \tag{4.7}$$

as an identity in $(\mathcal{U}_q(\mathfrak{m}(2)))^*$.

Proof. To prove (4.6) we test $b^R_{i,j}(t,s)$ against XY for arbitrary $X, Y\in\mathcal{U}_q(\mathfrak{m}(2))$. Then we use (4.1) and the homomorphism property of t^R, t^R being a representation of $\mathcal{U}_q(\mathfrak{m}(2))$. Develop $t^R(Y)f^{R,j}(s)$ in terms of the orthonormal basis $f^{R,k}(r)$ for a non-zero real r in order to find

$$\begin{aligned}\langle b^R_{i,j}(t,s), XY\rangle &= \sum_{k=-\infty}^{\infty} \langle t^R(X)f^{R,k}(r), f^{R,i}(t)\rangle\langle t^R(Y)f^{R,j}(s), f^{R,k}(r)\rangle \\ &= \sum_{k=-\infty}^{\infty} \langle b^R_{i,k}(t,r), X\rangle\langle b^R_{k,j}(r,s), Y\rangle\end{aligned}$$

which is what we wanted to prove.

We prove the first equality of (4.7), the other being proved analogously. Test the left hand side of (4.7) against an arbitrary element $X\in\mathcal{U}_q(\mathfrak{m}(2))$, use the duality to transpose the multiplication of two generalised matrix elements to the comultiplication on X and to transpose the $*$-operator, use (4.1) and the fact that t^R is a $*$-representation of $\mathcal{U}_q(\mathfrak{m}(2))$ to find

$$
(4.8)\quad \begin{cases} \left\langle \sum_{k=-\infty}^{\infty} b^R_{i,k}(t,s)(b^R_{j,k}(r,s))^*, X \right\rangle \\ = \left\langle \sum_{k=-\infty}^{\infty} b^R_{i,k}(t,s) \otimes (b^R_{j,k}(r,s))^*, \Delta(X) \right\rangle \\ = \sum_{k=-\infty}^{\infty} \sum_{(X)} \langle b^R_{i,k}(t,s), X_{(1)} \rangle \overline{\langle b^R_{j,k}(r,s), S(X_{(2)})^* \rangle} \\ = \sum_{k=-\infty}^{\infty} \sum_{(X)} \langle t^R(X_{(1)}) f^{R,k}(s), f^{R,i}(t) \rangle \langle t^R(S(X_{(2)})) f^{R,j}(r), f^{R,k}(s) \rangle \end{cases}
$$

with $\Delta(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}$. Next we use (4.1) to rewrite the right hand side in terms of the generalised matrix elements. By (4.6) we can recognise this as $\Delta(b^R_{i,j}(t,r))$. Using the duality we can transpose the comultiplication on such a generalised matrix element to multiplication on the other side. So (4.8) equals

$$
\sum_{k=-\infty}^{\infty} \sum_{(X)} \langle b^R_{i,k}(t,s), X_{(1)} \rangle \langle b^R_{k,j}(s,r), S(X_{(2)}) \rangle
$$

$$
= \langle \Delta(b^R_{i,j}(t,r)), (id \otimes S) \circ \Delta(X) \rangle = \langle b^R_{i,j}(t,r), m \circ (id \otimes S) \circ \Delta(X) \rangle,
$$

where $m(X \otimes Y) = XY$ denotes the multiplication in the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{m}(2))$. Next we use the Hopf algebra axiom $m \circ (id \otimes S) \circ \Delta(X) = \varepsilon(X)1$ as an identity in $\mathcal{U}_q(\mathfrak{m}(2))$, where $\varepsilon : \mathcal{U}_q(\mathfrak{m}(2)) \to \mathbf{C}$ denotes the counit of the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{m}(2))$, cf. [10, (3.4)], we obtain

$$
\varepsilon(X) \langle b^R_{i,j}(t,r), 1 \rangle = \varepsilon(X) \langle f^{R,i}(r), f^{R,j}(s) \rangle,
$$

which proves the first equality of (4.7), since the counit ε corresponds to the linear functional $1 \in (\mathcal{U}_q(\mathfrak{m}(2)))^*$. The other equality is proved analogously. □

Remark 4.3. It is possible to show that an identity similar to (4.4) already formally exists at the Hopf $*$-algebra level, i.e. as a non-polynomial identity involving elements of the Hopf $*$-algebra $\mathcal{A}_q(M(2)) \subset (\mathcal{U}_q(\mathfrak{m}(2)))^*$, cf. [10, § 3] for its definition. This can be done in an analogous fashion as in [12], [8], see also [15], [16]. To sketch this approach we consider elements $a \in (\mathcal{U}_q(\mathfrak{m}(2)))^*$ which satisfy

$$
(4.9)\qquad X_s.a = \lambda D.a \qquad \text{and} \qquad a.X_t = \mu a.D,
$$

for $s, t \in \mathbf{R}$ and some $\lambda, \mu \in \mathbf{C}$, where $X.a = \langle id \otimes X, \Delta(a) \rangle$ and $a.X = \langle X \otimes id, \Delta(a) \rangle$ and the duality is used in the second, respectively first factor, cf. [10, (3.8)]. It is easy to show that for non-zero a in the span of the matrix elements $t^R_{n,m}$, $n, m \in \mathbf{Z}$, the values of λ and μ are restricted to $s(q^{2j}-1)$ and $t(q^{2i}-1)$ for $i, j \in \mathbf{Z}$, cf. Proposition 3.1, cf. [8, § 2]. For such values of λ and μ the element a is determined up to a scalar; denote it by $\beta^R_{i,j}(t,s)$. This element is related to the generalised matrix element $b^R_{i,j}(t,s)$, cf. (4.1), by $D.\beta^R_{i,j}(t,s) = b^R_{i,j}(t,s)$. It should be noted that the element $b^R_{i,j}(t,s)$ is no longer an element of the Hopf $*$-algebra $\mathcal{A}_q(M(2))$. Neither it is contained in the algebra $\mathcal{A}_q^{ext}(M(2))$, which is an extension of $\mathcal{A}_q(M(2))$, cf. [10, § 3], unless $s = t = \infty$.

On the other hand, it follows from the Hopf $*$-algebra structure that the elements satisfying (4.9) for $\lambda = \mu = 0$ form a subalgebra of $\mathcal{A}_q(M(2))$, which we call the algebra of 'spherical elements'. Moreover, this algebra is generated by a single self-adjoint element $\rho_{s,t} \in \mathcal{A}_q(M(2))$, cf. [12, Proposition 4.7]. It is also easy to show that an element satisfying (4.9) when multiplied from the right by a spherical element again satisfies (4.9) for the same λ and μ, cf. [8, Proposition 2.2]. This motivates the search for a formal expression of the form $\beta^R_{i,j}(t,s) = \beta^{min}_{i,j}(t,s)f(\rho_{s,t})$ for some minimal element $\beta^{min}_{i,j}(t,s)$ satisfying (4.9) for the same λ and μ and some function f, cf. [8, Proposition 2.5]. This minimal element can be explicitly constructed by use of finite dimensional representations of $\mathcal{U}_q(\mathfrak{m}(2))$. An explicit expression for f up to a constant can be found using the action of the Casimir operator $BC \in \mathcal{U}_q(\mathfrak{m}(2))$, cf. [12, § 5]. However, the resulting identity is meaningless on the level of $\mathcal{A}_q(M(2))$ or $\mathcal{A}_q^{ext}(M(2))$. Furthermore, the constant in f can only be calculated explicitly by invoking the Graf addition formula (2.6) for the Jackson q-Bessel function.

Formally it is possible to give an explicit expression for certain Haar functionals, cf. [10, § 4], on formal power series in $\rho_{s,t}$ by means of an orthogonality relation, cf. [17] (and unpublished work by Koornwinder), similar to [12, Theorem 5.3]. The Schur orthogonality relation can be viewed as a q-analogue of the Hankel transform and contains as a special case the Hankel type orthogonality relations for the Hahn–Exton q-Bessel function, cf. [10, § 5], [13], in a way similar as the little q-Jacobi polynomials are contained in the Askey–Wilson polynomials, cf. [12, § 6].

To end this remark on this different formal approach we note that the ${}_2\varphi_1$-series in Proposition 4.1, respectively the orthogonality relations for such ${}_2\varphi_1$-series, can be formally obtained as a limit of the Askey–Wilson polynomials, respectively the orthogonality relations of the Askey–Wilson polynomials, cf. [1, §§ 1, 2], in a way similar to the limit transition of the Jacobi polynomials to the Bessel functions, respectively the limit transition of the orthogonality relations for the Jacobi polynomials to the Hankel transform.

Details of this formal approach can be found in [6, § 6].

5. A q-ANALOGUE OF GRAF'S ADDITION FORMULA

The observation that the right hand side of (4.4) is in fact a q-analogue of the Bessel function of shifted argument hints that the 'cohomomorphism property' (4.6) will lead to a q-analogue of Graf's addition formula in a similar way as in the classical case, cf. [20, 4.1.4(2)]. The goal of this section is to prove such an addition formula.

Proposition 5.1. *For* $n \in \mathbf{Z}$, $R > 0$, $r, s, t \in \mathbf{R}\backslash\{0\}$, $x, y \in \mathbf{C}\backslash\{0\}$ *satisfying* $|R^2q/s^2| < 1$, $|R^2q^{1+2n}/t^2| < 1$ *we have*

$$
(5.1)\quad \left\{\begin{aligned}
&\frac{(sy)^{-n}}{(-(R^2/s^2)q;q^2)_\infty}\,\frac{((s/t)xy,q^{2n+2};q^2)_\infty}{((s/t)xyq^{2n},q^2;s^2)_\infty}\\
&\quad\times {}_2\varphi_1\left(\begin{matrix}(t/s)xy,q^2(t/s)x^{-1}y^{-1}\\ q^{2n+2}\end{matrix};q^2,-\frac{R^2}{t^2}q^{1+2n}\right)\\
&=\sum_{k=-\infty}^{\infty}\frac{r^{-n}(R/r)^{2k}q^{-k^2+2k}}{(-(R^2/r^2)q^{1-2k};q^2)_\infty}\,\frac{((r/t)x,q^{2k+2n+2};q^2)_\infty}{(q^{2k+2n}(r/t)x,q^2;q^2)_\infty}\\
&\quad\times {}_2\varphi_1\left(\begin{matrix}(t/r)x,(t/r)q^2x^{-1}\\ q^{2k+2n+2}\end{matrix};q^2,-\frac{R^2}{t^2}q^{1+2n}\right)\\
&\quad\times\frac{((r/s)y,q^{2k+2};q^2)_\infty}{(q^{2k}(r/s)y,q^2;q^2)_\infty}\,{}_2\varphi_1\left(\begin{matrix}(s/r)y,(s/r)q^2y^{-1}\\ q^{2k+2}\end{matrix};q^2,-\frac{R^2}{s^2}q\right),
\end{aligned}\right.
$$

the sum being absolutely convergent, uniformly for x, y in compact sets not containing the origin.

Remark. (i) The result of Proposition 5.1 should be viewed as a q-analogue of Graf's addition formula for the Bessel function;

$$
(5.2)\quad \left(\frac{y-s^{-1}x}{y-sx}\right)^{n/2} J_n\left(\sqrt{(y-s^{-1}x)(y-sx)}\right)=\sum_{k=-\infty}^{\infty} s^k J_{n+k}(y)J_k(x),
$$

cf. [21, § 11.3]. Replace R by $(1-q^2)^2R^2$ and let $q\uparrow 1$ in (5.1) and use the q-gamma function (2.2) and $(-(1-q^2)^2R^2a;q)_\infty\to 1$ for any $a\in\mathbf{C}$ to obtain formally (5.2) with y, x and s replaced by

$$
2\frac{R}{t}\sqrt{\left(1-\frac{t}{r}x\right)\left(1-\frac{t}{r}x^{-1}\right)},\quad 2\frac{R}{s}\sqrt{\left(1-\frac{s}{r}y\right)\left(1-\frac{s}{r}y^{-1}\right)},
$$
$$
xy\sqrt{\frac{(1-(t/r)x^{-1})(1-(s/r)y^{-1})}{(1-(t/r)x)(1-(s/r)y)}}.
$$

(ii) Some known formulas for q-analogues of the Bessel functions can be obtained as special cases of (5.1). As a first example we see that, by letting $r\to\infty$ in (5.1), we obtain the Koornwinder–Swarttouw q-analogue of Graf's addition formula for the Jackson q-Bessel function, cf. (2.6). Here we use that

$$
J_\alpha^{(2)}(2z;q)=(-z^2;q)_\infty\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}z^\alpha\,{}_2\varphi_1\left(\begin{matrix}0,0\\ q^{\alpha+1}\end{matrix};q,-z^2\right),\quad |z|<1,
$$

by Heine's transformation formula [2, (1.4.6)].

Secondly, we show how to obtain the Hansen–Lommel orthogonality relations for the Jackson q-Bessel function from (5.1). Choose $s=t$ and replace xy by q^{-1}, then the left hand side of (5.1) reduces to $\delta_{n,0}(-R^2q/t^2;q^2)_\infty^{-1}$. If we let $r\to\infty$, we find the Hansen–Lommel orthogonality relation for the Jackson q-Bessel function, cf. [7, Theorem 3.1] or the first equality of (3.10), whereas if we let $t\to\infty$, we obtain the dual Hansen–Lommel orthogonality relation for the Jackson q-Bessel function, cf. [7, Theorem 3.3] or the second equality of (3.10).

Note that all these special cases have played an important part in deriving the q-analogue of Graf's addition formula (5.1).

(iii) If we put $x = st^{-1}y^{-1}$ in (5.1) the left hand side reduces to $\delta_{n,0}(-R^2q/s^2;q^2)_\infty^{-1}$, so that we obtain Hansen–Lommel type biorthogonality relations for this kind of q-Bessel function. The resulting biorthogonality relations can also be obtained from the unitarity property (4.7) for the generalised matrix elements for $r = s$.

Proof. Consider (4.6) for $i = -n$ and $j = 0$ and test it against $A^\lambda \otimes A^\mu$ for $\lambda, \mu \in \mathbf{Z}$. For the left hand side we use

$$\langle \Delta(b^R_{-n,0}(t,s)), A^\lambda \otimes A^\mu \rangle = \langle b^R_{-n,0}(t,s), A^{\lambda+\mu} \rangle$$

for which we use Proposition 4.1 in combination with (4.5) to get an explicit expression in terms of a ${}_2\varphi_1$-series. For $\langle b^R_{-n,k}(t,r), A^\lambda \rangle$ we use Proposition 4.1 in combination with (4.5) as well and for $\langle b^R_{k,0}(r,s), A^\mu \rangle$ we use Proposition 4.1. A straightforward calculation proves (5.1) for $x = q^\lambda$ and $y = q^\mu$, $\lambda, \mu \in \mathbf{Z}$.

Next we show that the sum over k in (5.1) is absolutely convergent and uniformly convergent for x and y in compact sets. For $|z| < 1$ and $n \in \mathbf{Z}$ we have the estimate

$$\begin{aligned}
&\left| (q^{n+1};q)_\infty \, {}_2\varphi_1\left({a, b \atop q^{n+1}} ; q, z \right) \right| \\
&\leq \sum_{k=\max(0,-n)}^{\infty} \left| \frac{(q^{n+k+1};q)_\infty}{(q;q)_k} (a,b;q)_k z^k \right| \\
&\leq (-q^{\max(0,n)+1}, -|a|, -|b|; q)_\infty \sum_{k=0}^{\infty} \frac{|z|^k}{(q;q)_k} \\
&= \frac{(-q^{\max(0,n)+1}, -|a|, -|b|; q)_\infty}{(|z|;q)_\infty}
\end{aligned}$$

by the q-binomial theorem, cf. [2, (1.3.15)]. By use of this estimate it follows that the sum in (5.1) for $|-R^2q^{1+2n}/s^2| < 1$ and $|-R^2q/t^2| < 1$ is absolutely convergent and the convergence is uniform in x and y on compact sets not containing the origin. Note that the apparent singularities in the right hand side of (5.1) at $x = q^{-z}t/r$, $y = q^{-z}s/r$ for z in $\mathbf{N}$ are removable by (4.5).

To prove the full formula (5.1) we extend the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{m}(2))$ to incorporate elements of the form A^λ for $\lambda \in \mathbf{C}$. To do this we introduce linear functionals, denoted by $A^\lambda B^r C^s$, $\lambda \in \mathbf{C}$, $r, s \in \mathbf{Z}_+$, on $\mathcal{A}_q(\tilde{M}(2))$ as in [10, Propsition 3.2] with $p \in \mathbf{Z}$ replaced by $\lambda \in \mathbf{C}$, which yields well-defined linear functionals. The set of these linear functionals on $\mathcal{A}_q(\tilde{M}(2))$ forms a Hopf $*$-algebra $\mathcal{U}_q^c$ by transposing the Hopf $*$-algebra structure of $\mathcal{A}_q(\tilde{M}(2))$ given in [10, § 3] using dual Hopf $*$-algebras, cf. [19]. We find the following relations for the generators A^λ, $\lambda \in \mathbf{C}$, B and C;

$$\begin{aligned}
&A^\lambda B = q^\lambda B A^\lambda, \qquad A^\lambda C = q^{-\lambda} C A^\lambda, \qquad BC = CB, \\
&A^\lambda A^\mu = A^{\lambda+\mu}, \qquad A^0 = 1,
\end{aligned}$$

cf. (3.1). The $*$-operator, cf. (3.2), is then given by

$$(A^{\lambda})^* = A^{\bar{\lambda}}, \qquad B^* = C, \qquad C^* = B.$$

For the comultiplication we obtain $\Delta(A^{\lambda}) = A^{\lambda} \otimes A^{\lambda}$, $\lambda \in \mathbf{C}$.

The $*$-representation t^R of $\mathcal{U}_q(\mathrm{m}(2))$ is extended to $\mathcal{U}_q^c$ by defining $t^R(A^{\lambda})e_n = q^{n\lambda}e_n$, cf. (3.3). So $\langle t^R_{n,m}, A^{\lambda}\rangle = \delta_{n,m}\, q^{n\lambda}$ and Proposition 4.1 remains valid for $\lambda \in \mathbf{C}$. Thus the proof of Proposition 5.1 goes through as well for $\lambda, \mu \in \mathbf{C}$ and after replacing $x = q^{\lambda}$, $y = q^{\mu}$, $\lambda, \mu \in \mathbf{C}$, we obtain (5.1) in full generality. $\square$

Remark. (i) It does not seem possible to prove (5.1) in full generality from the case $x = q^{\lambda}$, $y = q^{\mu}$, for $\lambda, \mu \in \mathbf{Z}$, by analytic function theoretic means, such as analytic continuation or Carlson's theorem.
(ii) The general formula (5.1) does not lead to a more general addition formula if not restricted to the case $j = 0$.

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